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EFFECT OF ANOMALOUS DISPERSION DEPENDENCES ON SCATTERING AND GENERATION OF INTERNAL WAVES

I. V. Sturova

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The stable stratification of fluids, corresponding to an increasing density in the direction of the gravity force, results in the existence of internal waves. Low amplitude waves are described by the linear theory of internal waves, whose basis is described in detail, for example, in [1]. For a horizontally homogeneous fluid at rest there exists a countable number of free harmonic wave modes when a continuous density variation occurs only at a finite interval of depths. To each wave mode there corresponds a dispersion dependence of the wave frequency on the wave number, making it possible to determine the phase and group velocities of the given wave. The density distributions with normal dispersion characteristics are assumed to be those, for which each wave mode has a group velocity decreasing monotonically with increasing wave number. In the opposite case, when along the maximum value of the group velocity for given internal waves there occur local maxima with nonvanishing wave number values, the dispersion dependences are called anomalous.

An investigation of the nature of density distributions, for which anomalous dispersion dependences are possible, was presented in [2]. It was shown, in particular, that anomalous dispersion dependences can be generated for those density distributions, in which there exists at least one waveguide portion with a buoyancy frequency being different from the maximum frequency, nonvanishing, and slowly varying over a wavelength of vertical standing eigenoscillations corresponding to it. An example of a fluid with anomalous dispersion properties can be a three-layered fluid with linearly stratified vortices and mid-layers with a homogeneous lower layer of infinite depth.

It is assumed that the inviscid, incompressible liquid occupies the region $-\infty < x < \infty$, $-\infty < y < H$ (x is the horizontal, and y the vertical coordinate). In the unperturbed state the density distribution is

$$\rho(y) = \begin{cases} \rho_1[1 - \alpha_1(y - H)] & (H_2 < y < H), \\ \rho_1(1 + \alpha_1 H_1)[1 - \alpha_2(y - H_2)] & (0 < y < H_2), \\ \rho_2 = \rho_1(1 + \alpha_1 H_1)(1 + \alpha_2 H_2) & (y < 0) \end{cases}$$

(H_1 and H_2 are the thicknesses of the upper and mid-layer, and $H = H_1 + H_2$). For definiteness we put $\alpha_2 > \alpha_1 > 0$, i.e., the density gradient is maximum in the mid-layer.

In the present study we investigate the scattering of an internal wave, incident on a solid horizontal elliptic cylinder, and determine the wave resistance when a uniform flow bypasses such a body. The cylinder axis is parallel to the front of the incident wave and perpendicular to the velocity of the incident flow, so that the problems considered are two-dimensional. For simplicity it is assumed that the cylinder is totally immersed in the lowest layer, the flow in which being treated as a potential flow, and that the upper layer is bounded by a solid lid. The internal wave equations are described within the Boussinesq approximation. In both problems it is assumed that the body is sufficiently deeply immersed under the separation boundary between the mid- and lower layer. We compare the characteristics of wave motion in fluids with normal and anomalous dispersion properties.

1. In the diffraction problem the equations of wave motion for the vertical velocity v(x, y, t) in the upper and mid-layers are

$$\partial^2 \Delta v_1/\partial t^2 + N_1^2 \partial^2 v_1/\partial x^2 = 0 \quad (H_2 < y < H),$$
$$\partial^2 \Delta v_2/\partial t^2 + N_2^2 \partial^2 v_2/\partial x^2 = 0 \quad (0 < y < H_2),$$

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where the subscripts 1 and 2 refer, respectively, to the upper and mid-layers, $N_{1,2} = \sqrt{g\alpha_{1,2}}$, g is the gravity force acceleration, and t is time. In the lower homogeneous layer the total velocity potential $\varphi(x, y, t)$ satisfies the equation

$$\Delta\varphi = 0 \quad (y < 0). \tag{1.1}$$

The boundary conditions are the following:

$$\begin{aligned}
&v_1 = 0 && (y = H),\\
&v_1 = v_2,\ \ \partial v_1/\partial y = \partial v_2/\partial y && (y = H_2),\\
&v_2 = \partial\varphi/\partial y,\ \ \partial v_2/\partial y = \partial^2\varphi/\partial y^2 && (y = 0),\\
&\partial\varphi/\partial y \to 0 && (y \to -\infty).
\end{aligned} \tag{1.2}$$

It is known from the theory of linear internal waves [1] that in such a fluid the existence of free internal waves is possible only with $\omega < N_2$. The wave incident from the left can be an arbitrary internal mode with a vertical velocity

$$v = \mathrm{Re}W(y)\exp[i(kx - \omega t)].$$

The wave number k satisfies the dispersion relation

$$\mathrm{tg}\,\gamma_2 H_2 = \gamma_2 \cdot \begin{cases} \dfrac{\gamma_1 + k\,\mathrm{tg}\,\gamma_1 H_1}{\gamma_2^2\,\mathrm{tg}\,\gamma_1 H_1 - k\gamma_1} & (\omega < N_1),\\[2ex] \dfrac{\beta_1 + k\,\mathrm{th}\,\beta_1 H_1}{\gamma_2^2\,\mathrm{th}\,\beta_1 H_1 - k\beta_1} & (\omega > N_1) \end{cases} \tag{1.3}$$

($\beta_1 = k\sqrt{1 - N_1^2/\omega^2}$, $\gamma_{1,2} = k\sqrt{N_{1,2}^2/\omega^2 - 1}$). There exists a countable number of values k_j ($k_1 < k_2 < \ldots$), satisfying the given dispersion relation. The eigenfunctions $W(y, k_j)$ of the wave modes are represented in the form

$$W(y, k) = \frac{k}{\gamma_2} \cdot \begin{cases} (k\sin\gamma_2 H_2 + \gamma_2\cos\gamma_2 H_2)A(k, y) & (H_2 < y < H),\\ k\sin\gamma_2 y + \gamma_2\cos\gamma_2 y & (0 < y < H_2),\\ \gamma_2 e^{ky} & (y < 0), \end{cases}$$

where

$$A(k,y) = \begin{cases} \sin\gamma_1(H - y)/\sin\gamma_1 H_1 & (\omega < N_1),\\ \mathrm{sh}\,\beta_1(H - y)/\mathrm{sh}\,\beta_1 H_1 & (\omega > N_1). \end{cases}$$

The eigenfunctions are orthogonal:

$$N_1^2\int_{H_2}^{H} W(y, k_j)W(y, k_m)dy + N_2^2\int_{0}^{H_2} W(y, k_j)W(y, k_m)dy = \begin{cases} c_j^2 & (m = j),\\ 0 & (m \neq j). \end{cases}$$

The quantity c_j can be considered to be an energy characteristic of the given internal mode. In what follows the wave scattering parameters are determined within this energy normalization.

The potential of perturbed motion in the lower layer is represented as

$$\varphi = \mathrm{Re}(\Phi_0 + \Psi)\exp(i\omega t).$$

Here the incident wave potential $\Phi_0(x, y)$ is

$$\Phi_0 = c_j\varphi_0,\ \ \varphi_0 = \exp[k_j(y - ix)],$$

and the diffraction potential $\Psi(x, y)$ satisfies the nonleaking condition at the surface of the cylinder L

$$\partial\Psi/\partial n = -\partial\Phi_0/\partial n \quad (x, y \in L)$$

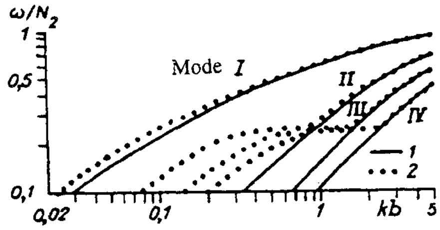

Fig. 1

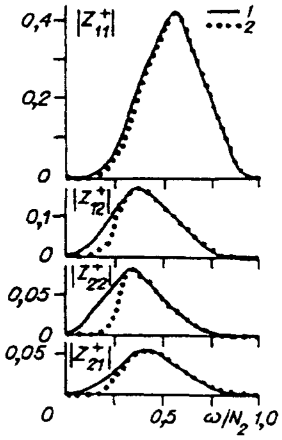

Fig. 2

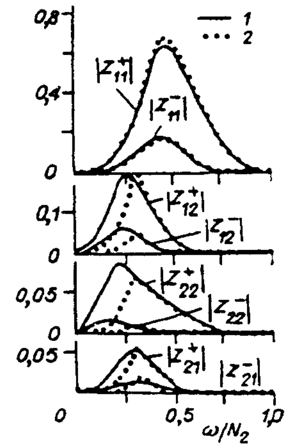

Fig. 3

($\mathbf{n} = (n_x, n_y)$ is the internal normal to the surface of the body). The radiation conditions must be satisfied in the given field, implying that only waves outgoing from the body are formed during the scattering.

The analysis of the given problem is conveniently performed by means of the Green's function (for more detail see [3]). The Green's function in the lower layer $G(x, y, \xi, \eta)$ is

$$G = \ln(rr_1) + 2\,\mathrm{pv}\int_0^\infty e^{k(y+\eta)}\cos k(x-\xi)D(k)dk - 2i\pi\sum_{j=1}^\infty P_j e^{k_j(y+\eta)}\cos k_j(x-\xi), \tag{1.4}$$

where

$$D(k) = \begin{cases} T(k, \gamma_1, \gamma_2) & (\omega < N_1), \\ T(k, i\beta_1, \gamma_2) & (\omega > N_1); \end{cases} \quad T(k, \gamma_1, \gamma_2) = \frac{T_1(\gamma_1, \gamma_2)}{T_2(k, \gamma_1, \gamma_2)};$$

$$T_1(\gamma_1, \gamma_2) = \gamma_2 \operatorname{tg}\gamma_1 H_1 + \gamma_1 \operatorname{tg}\gamma_2 H_2;$$

$$T_2(k, \gamma_1, \gamma_2) = kT_1(\gamma_1, \gamma_2) + \gamma_2(\gamma_1 - \gamma_2 \operatorname{tg}\gamma_1 H_1 \operatorname{tg}\gamma_2 H_2);$$

$$P_j = (T_1/T_2')|_{k=k_j};$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2; \quad r_1^2 = (x-\xi)^2 + (y+\eta)^2;$$

The symbol pv denotes an integral in the principal value sense, and the prime denotes differentiation with respect to k.

From analyzing the behavior of G at $|x| \to \infty$ it is seen that for continuous stratification each incident mode is scattered into an infinite number of modes. As a result, the solutions of the diffraction problem can be determined by forward (+) and backward (−) scattering matrices $\|Z_{jm}^{\pm}\|$, where for each element the row number j corresponds to the incident mode number, and the column number m — to the number of the scattered internal wave:

$$Z_{jm}^{\pm} = a_{jm}^{\pm}c_m/c_j.$$

Here

$$a_{jm}^{\pm} = -iP_m V^{\pm}(k_m, k_j);$$

$$V^{\pm}(k, k_s) = \int_L e^{k(\eta \pm i\xi)}\left[\frac{\partial\varphi_0}{\partial n} + \frac{k_s}{c_s}(n_\eta \pm i n_\xi)\Psi\right]dl.$$

TABLE 1

j	m = 1	2	3	4	5	6	7	8	9	10
	$10^3 \cdot \lvert z^+_{jm} \rvert$									
1	35	41	46	97	125	51	31	44	53	19
2	8	10	11	24	31	13	8	11	14	5
3	5	6	6	14	18	8	5	7	9	3
4	7	9	11	24	31	13	9	14	17	7
5	12	15	17	39	52	22	15	23	30	12
	$10^3 \cdot \lvert z^-_{jm} \rvert$									
1	12	14	15	31	40	16	10	14	16	6
2	3	3	3	7	9	4	2	3	3	1
3	2	2	2	4	5	2	1	1	2	1
4	2	3	3	6	7	3	2	2	2	1
5	4	4	5	9	11	4	2	3	3	1

TABLE 2

j	m = 1	2	3	4	5	6	7	8
	$10^3 \cdot \lvert z^+_{jm} \rvert$							
1	67	152	61	8	4	2	0	0
2	29	80	44	14	2	0	0	0
3	6	24	21	12	6	2	1	0
4	1	5	9	8	6	3	1	0
5	0	1	3	4	4	3	1	1
	$10^3 \cdot \lvert z^-_{jm} \rvert$							
1	22	48	18	2	1	1	—	—
2	9	16	4	1	1	0	—	—
3	2	2	0	0	0	0	—	—

The complex values Z_{jj}^+ and Z_{jj}^- are, respectively, the transmission and reflection coefficients of the j-th internal mode.

Within the deeply immersed body approximation the diffraction potential is determined for an unbounded homogeneous fluid without account of stratification. In this case the nonleaking conditions are satisfied on the body contour, as well as the damping condition far from it. An application of this method to a two-layer fluid and a comparison of the approximate solution with the numerical solution of the total problem are presented in [3].

For an elliptic contour, given by the equation

$$x^2/a^2 + (y + h)^2/b^2 = 1$$

(a and b are the major and minor semiaxes, and h is the depth of the immersed center of the ellipse), expressions for $V^{\pm}$ in the approximate solution are

$$V^+(k, k_s) = 4\pi \sum_{n=1}^{\infty} n \left(\frac{a+b}{a-b}\right)^n J_n(kc) J_n(k_s c),$$

$$V^-(k, k_s) = \frac{2\pi c k k_s}{k + k_s} [J_1(kc) J_0(k_s c) + J_0(kc) J_1(k_s c)],$$

where $c = \sqrt{a^2 - b^2}$, and J_n is the Bessel function of the first kind of order n.

In the special case of a circular cylinder of radius b

$$V^+(k, k_s) = 4\pi b \sqrt{k k_s} J_1(2b\sqrt{k k_s}), \quad V^-(k, k_s) \equiv 0$$

TABLE 3

j	m									
	1	2	3	4	5	1	2	3	4	5
	Case 1					Case 2				
	$10^3 \cdot \lvert z^+_{jm} \rvert$									
1	<u>528</u>	94	6	0	0	<u>534</u>	100	7	0	0
2	31	<u>46</u>	14	1	0	33	<u>49</u>	14	1	0
3	1	8	<u>9</u>	3	1	1	8	<u>9</u>	3	1
4	—	—	—	—	—	0	1	2	<u>2</u>	0
	$10^3 \cdot \lvert z^-_{jm} \rvert$									
1	<u>159</u>	11	2	—	—	<u>164</u>	15	2	—	—
2	4	<u>1</u>	0	—	—	5	<u>1</u>	0	—	—

TABLE 4

j	$U_j/\sqrt{ga}$		j	$U_j/\sqrt{ga}$	
	Case 1	Case 2		Case 1	Case 2
1	0,5523	0,6605	6	0,0078	0,0414
2	0,0388	0,1782	7	0,0065	0,0368
3	0,0195	0,0963	8	0,0056	0,0320
4	0,0130	0,0655	9	0,0049	0,0277
5	0,0097	0,0499	10	0,0043	0,0244

(I_1 is the modified Bessel function of the first order). Note that waves scattered backward are always absent for a circular cylinder, placed in a fluid layer of infinite depth.

Numerical calculations were carried out for $H_1/b = 1$, $H_2/b = 20$ for two cases of stratification: 1) $N_1 = 0$, 2) $\varepsilon = N_1/N_2 = 0.25$. The dispersion curves for the first four modes are presented in Fig. 1 by curves 1, 2. In case 1 the stratified fluid has only one waveguide layer with a constant value of a buoyancy frequency. In case 2 there exists a substantially thick upper layer with a substantial buoyancy frequency, nonvanishing and differing from the maximum value. The dispersion modes for the first mode are close in these cases, but for higher modes one observes in case 2 sharply expressed "bands" at $\omega = N_1$, whose appearance also gives rise to nonmonotonic behavior of the group velocities.

For a circular cylinder with $h/b = 1.5$ the transmission coefficients for the first two modes, as well as the scattering coefficients of the first mode in the second $|Z_{12}^+|$ and, conversely, of the second mode in the first $|Z_{21}^+|$, are presented in Fig. 2. For an elliptic cylinder ($a/b = 2$, $h/b = 1.5$) the similar characteristics for the forward and backward scattering events are given in Fig. 3. Curves 1, 2 in Figs. 2, 3 show the results for the corresponding cases. It is seen that the transmission and reflection coefficients for the first mode are practically independent of N_1, while the other characteristics in the region $0 < \omega/N_2 < 0.25$ differ sharply for these two types of stratification. The coefficients $|Z_{12}^{\pm}|$, $|Z_{22}^{\pm}|$, $|Z_{21}^{\pm}|$, are substantially smaller in case 2 than in case 1. This is, obviously, explained by the fact that in case 2 there is substantial scattering in the higher modes. The scattering matrices are presented in Tables 1-3. Table 1 shows the values of $10^3 \cdot |Z_{jm}^{\pm}|$ ($j = \overline{1,5}$, $m = \overline{1,10}$) for the elliptic contour with $\omega/N_2 = 0.2$ for case 2. The underlined diagonal elements correspond to transmission and reflection coefficients for the corresponding modes. Table 2 provides the similar results for case 1. The smaller size of this table is explained by the fact that in case 1 the scattering for the higher modes is insignificantly small. It is seen that for nonmonotonic variation of group velocities more intense scattering is possible in the higher modes. This is observed only for $\omega < N_1$. At higher frequencies the incident waves provide practically identical results in both cases. In Table 3 we show the analogous coefficients for $\omega/N_2 = 0.4$. When the frequency of the incident wave tends to the limiting value N_2 wave scattering ceases.

Similarly one can also solve the diffraction problem of excitation of internal waves for steady-state body oscillations: horizontal, vertical, and rotational. The Green's functions of these problems coincide.

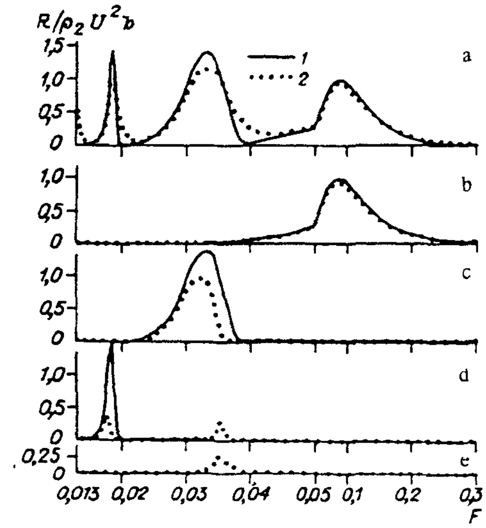

Fig. 4

2. For stationary uniform flow past an immersed body with velocity U in the direction of the negative x axis the equations of wave motion for the vertical velocity v(x, y) are

$$\Delta v_1 + \varepsilon^2\nu^2 v_1 = 0 \quad (H_2 < y < H),$$
$$\Delta v_2 + \nu^2 v_2 = 0 \quad (0 < y < H_2),$$

where $\nu = N_2/U$. In the lower layer the velocity potential of the wave motion $\varphi(x, y)$ satisfies Eq. (1.1). The boundary conditions of the stationary problem coincide with (1.2). According to the radiation condition, in the far field the wave motion can be substantial only behind the body.

One of the fundamental difficulties of solving this problem is satisfaction of the exact continuity condition on the body surface

$$\partial\varphi / \partial n = Un_x \quad (x, y \in L).$$

Most studies involved with the determination of internal wave characteristics were carried out by replacing the real finite body by a system of point singularities, similarly to the way this is done for a homogeneous unbounded body (see the review [4]). Application of numerical methods makes it possible to solve the problem considered within the full statement of the problem and, in particular, to determine the wave load acting on the immersed body. An example of comparing numerical results with an approximate solution for the wave resistance of an elliptic cylinder in a two-layer unbounded fluid is presented in [5]. It has been shown that the dipole approximation provides a quite satisfactory upper bound estimate for the wave resistance.

The Green's function of the given problem in the lower layer is, similarly to (1.4):

$$G = \ln(rr_1) + 2\text{p}\nu\int_0^\infty e^{k(y+\eta)} \cos k(x - \xi)S(k)dk + 2\pi\sum_{j=1}^{M} P_j e^{k_j(y+\eta)} \cos k_j(x - \xi). \tag{2.1}$$

Here

$$S(k) = \begin{cases} T(k, \gamma_1, \gamma_2) & (k < \varepsilon\nu), \\ T(k, i\beta_1, \gamma_2) & (\varepsilon\nu < k < \nu), \\ T(k, i\beta_1, i\beta_2) & (k > \nu); \end{cases}$$
$$\beta_2 = \sqrt{k^2 - N_2^2/U^2}.$$

The functions T and P_1 coincide with the expressions given in (1.4), in which the replacement $\omega = kU$ was performed. The same replacement must also be made in (1.3) to determine k_j. The integrand expression in (2.1) can have simple bands when $k < \nu$, whose number M increases with increasing ν. We denote by $\nu_m(\nu_1 < \nu_2 < \ldots)$ the ν values, for which roots k_m are created. The ν_m values are determined by solving the equation tg $\nu_m H_2 \cdot$tg $\varepsilon\nu_m H_1 = \varepsilon$. For a given ν the value of M is found from the inequality $\nu_M < \nu < \nu_{M+1}$.

Wave motion is generated behind the body only when $U < U_1 = N_2/\nu_1$ and is the sum of M harmonic waves. The number of excited waves increases with decreasing velocity of flow past the body.

The dipole approximation for the wave resistance R is discussed in [6], and for an elliptic cylinder in the three-layer fluid considered is

$$R = \rho_2 d^2 \sum_{j=1}^{M} \frac{k_j^2 e^{-2k_j h}}{Q_1(k_j) + Q_2(k_j)},$$

where

$$Q_1(k) = \frac{(k \sin\gamma_2 H_2 + \gamma_2 \cos\gamma_2 H_2)^2}{2\gamma_2^2} \begin{cases} \dfrac{2\gamma_1 H_1 - \sin 2\gamma_2 H_2}{\gamma_1 \sin^2 \gamma_1 H_1} & (k < \varepsilon\nu), \\ \dfrac{\operatorname{sh} 2\beta_1 H_1 - 2\beta_1 H_1}{\beta_1 \operatorname{sh}^2 \beta_1 H_1} & (k > \varepsilon\nu); \end{cases};$$

$$Q_2(k) = H_1 + \frac{1}{k} + \frac{\sin 2\gamma_2 H_2}{2\gamma_2} - \frac{2k}{\gamma_2}\sin^2\gamma_2 H_2 + \frac{k^2}{\gamma_2^2}\left(H_2 - \frac{\sin 2\gamma_2 H_2}{2\gamma_2}\right);$$

and $d = \pi b(a + b)U$ is the dipole moment for the elliptic contour.

Calculations were carried out for the fluid stratification cases considered in Section 1 and for the body geometry $N_2{}^2H_2/g = 0.03$. The critical velocity values of the first 10 modes are given in Table 4. It is seen that the difference between the two cases investigated becomes very large with increasing mode number. The behavior of the wave resistance is presented in Fig. 4, where curves 1, 2 show the corresponding cases of fluid stratification. Figure 4 shows the total result, while Figs. 4b-d show the contributions of the first four modes. It is seen that despite the sharp differences between the critical velocities, the behavior of the total wave resistance does not differ very strongly in these two cases. The mode expansion of the wave resistance shows that the contributions of the first modes are practically equal due to the adjacency of their dispersion dependences, while the contributions of higher modes differ substantially. For normal dispersion dependences the wave resistance has one maximum for each wave mode, while for anomalous ones their number increases with mode number. It is interesting to note that the locations of maximum appearances for anomalous dispersion dependences are near the velocity values where maxima are generated during normal behavior of the dispersion dependences. Thus, for $F \equiv U/\sqrt{ga} = 0.035$ in case 1 the wave resistance is determined by the second mode, while in case 2 it is determined by the total contribution of modes with $j = \overline{2,6}$, for $F = 0.0175$ in case 1 — by the third mode, and in case 2 — by the contribution of modes with $j = \overline{3,10}$. Figure 4 shows only the range of values $0.013 \le F \le 0.3$, where there exist no more than 3 modes in case 1 and no more than 18 modes in case 2.

Thus, it has been shown in this study that anomalous dispersion dependences can lead to a "leapfrog" of internal wave modes, when the excitation of higher modes occurs more intensely than that of the lower modes.

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